Partial Fractions

Extra Credit

1. $$\frac{x-9}{x^2-3x-18}=\frac{x-9}{(x-6)(x+3)}=\frac{-3/9}{x-6}+\frac{-12/-9}{x+3}=\frac{1}{3}\left(\frac{4}{x+3}-\frac{1}{x-6}\right)\quad \int \Rightarrow \frac{1}{3}\ln\left[K\cdot\frac{(x+3)^4}{x-6}\right]$$

2. $$\frac{x^2-3}{(x-0)(x-4)(x+1)}=\frac{-3/(-4)(1)}{x-0}+\frac{13/(4)(5)}{x-4}+\frac{-2/(-1)(-5)}{x+1}=\frac{3}{4x}+\frac{1}{20}\left(\frac{13}{x-4}-\frac{8}{x+1}\right)$$

$$=\frac{1}{20}\left(\frac{15}{x}+\frac{13}{x-4}-\frac{8}{x+1}\right)\quad \int \Rightarrow \frac{1}{20}\ln\left[K\cdot\frac{x^{15}(x-4)^{13}}{(x+1)^8}\right]$$

3. $$\frac{8}{x^2+4x-12}=\frac{8}{(x+6)(x-2)}=\frac{8/-8}{x+6}+\frac{8/8}{x-2}=\frac{1}{x-2}-\frac{1}{x+6}\quad \int \Rightarrow \ln\left[K\cdot\frac{x-2}{x+6}\right]$$

4. $$\frac{2x}{x^2-9}=\frac{-6/-6}{x+3}+\frac{6/6}{x-3}=\frac{1}{x+3}+\frac{1}{x-3}\quad \int \Rightarrow \ln\left[K\cdot\left(x^2-9\right)\right]$$

5. $$\frac{x}{(x-6)(x+2)^2}=\frac{A}{x-6}+\frac{B}{x+2}+\frac{C}{(x+2)^2}=\frac{A(x+2)^2+B(x-6)(x+2)+C(x-6)}{(x-6)(x+2)^2}$$

$x=6 \Rightarrow A=6/64=3/32,\quad x=-2 \Rightarrow C=-2/-8=1/4,\ x=0 \Rightarrow -9/4-12B=0 \Rightarrow B=-3/32$

$$\Rightarrow \frac{3}{32}\left(\frac{1}{x-6}-\frac{1}{x+2}\right)+\frac{1}{4}\cdot\frac{1}{(x+2)^2}\quad \int \Rightarrow \frac{3}{32}\ln\left[K\cdot\frac{x-6}{x+2}\right]-\frac{1}{4}\cdot\frac{1}{x+2}$$

6. $$\frac{2}{x^3(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x+1}\quad \Rightarrow \left(Ax^2+Bx+C\right)(x+1)+Dx^3=2$$

$x=0 \Rightarrow C=2,\quad x=-1 \Rightarrow D=-2$ now take derivative to use repeated root

$$\frac{d}{dx} \Rightarrow (2Ax+B)(x+1)+Ax^2+Bx+C+3Dx^2=0$$

$x=0 \Rightarrow B=-C=-2,\quad x=-1 \Rightarrow A-B+C+3D=0 \Rightarrow A=2$

$$\Rightarrow \frac{2}{x}-\frac{2}{x+1}-\frac{2}{x^2}+\frac{2}{x^3}\quad \int \Rightarrow \ln\left[\frac{x^2}{(x+1)^2}\right]+\frac{2x-1}{x^2}+C$$

7. $\dfrac{2}{x\left(x^2+1\right)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1} \quad \Rightarrow \quad A(x^2+1)+(Bx+C)x=2$

$x=0 \Rightarrow A=2, \quad x=i \Rightarrow Bi^2+Ci=2 \Rightarrow -B+Ci=2+0i \Rightarrow B=-2, C=0$

$\Rightarrow \dfrac{2}{x}-\dfrac{2x}{x^2+1} \qquad\qquad \int \Rightarrow \ln\left[\dfrac{x^2}{x^2+1}\right]+C$

8. $\dfrac{2}{x^2\left(x^2+1\right)^2}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+1}+\dfrac{Ex+F}{\left(x^2+1\right)^2}$

$\Rightarrow (Ax+B)\left(x^2+1\right)^2+\left[(Cx+D)(x^2+1)+(Ex+F)\right]x^2=2$

$x=0 \Rightarrow B=2, \quad x=i \Rightarrow -Ei-F=2 \Rightarrow F=-2, E=0$

$\dfrac{d}{dx}: \; A\left(x^2+1\right)^2+(4Ax^2+4Bx)(x^2+1)+(3Cx^2+2Dx)(x^2+1)+2x(Cx^3+Dx^2)+3Ex^2+2Fx=0$

$x=0 \Rightarrow A=0, \quad x=i \Rightarrow 2C-2Di-4i=0 \Rightarrow C=0, D=-2$

Check: $\dfrac{2}{x^2}+\dfrac{-2}{x^2+1}+\dfrac{-2}{\left(x^2+1\right)^2}=\dfrac{2}{x^2}+\dfrac{-2x^2-4}{\left(x^2+1\right)^2}=\dfrac{2x^4+4x^2+2-2x^4-4x^2}{x^2\left(x^2+1\right)^2}=\dfrac{2}{x^2\left(x^2+1\right)^2}$

$\int \Rightarrow \dfrac{-2}{x}-3\tan^{-1}(x)-\dfrac{x}{x^2+1}+C$

9. $\dfrac{x^2+1}{x^3+9x}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+9}=\dfrac{A(x^2+9)+(Bx^2+Cx)}{x^3+9x}=\dfrac{1}{9}\left(\dfrac{1}{x}+\dfrac{8x}{x^2+9}\right)$

$\int \Rightarrow \dfrac{1}{9}\ln\left[Kx\cdot\left(x^2+9\right)^4\right]$

10. $\dfrac{x^2}{(x-1)\left(x^2+4x+5\right)}=\dfrac{x^2}{(x-1)\left[(x+2)^2+1\right]}=\dfrac{A}{x-1}+\dfrac{B(x+2)+C}{(x+2)^2+1}$

$\Rightarrow A\left[(x+2)^2+1\right]+\left[B(x+2)+C\right](x-1)=x^2$

$x=1 \Rightarrow A\left[9+1\right]=1 \Rightarrow A=1/10$

$x=-2+i \Rightarrow [Bi+C](-3+i)=(-2+i)^2 \Rightarrow (-3C-B)+(-3B+C)i=3-4i$

$\Rightarrow \begin{Bmatrix} B+3C=-3 \\ 3B-C=4 \end{Bmatrix} \Rightarrow \begin{pmatrix} B \\ C \end{pmatrix}=\dfrac{1}{-10}\begin{pmatrix} -1 & -3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \Rightarrow B=1/10, C=5/10$

$\therefore \dfrac{x^2}{(x-1)\left(x^2+4x+5\right)}=\dfrac{1}{10}\left(\dfrac{1}{x-1}+\dfrac{x+7}{x^2+4x+5}\right)=\dfrac{1}{10}\left(\dfrac{1}{x-1}+\dfrac{(x+2)+5}{(x+2)^2+1}\right)$

$\int \Rightarrow \dfrac{1}{10}\ln\left[K(x-1)\cdot\sqrt{x^2+4x+5}\right]+\dfrac{1}{2}\tan^{-1}(x+2)$