Analysis Qualifying Exam

January 2016

Please solve any 5 of the following 8 problems. Be sure to write clearly and give sufficient explanations. Solving more than 5 will not result in extra points.

Problem 1

(a) State the Monotone Convergence Theorem (MCT).

(b) State Fatou's Lemma.

(c) Use Fatou's Lemma to prove MCT.

Problem 2

Let $\{f_n\}_{n=1}^{\infty}$, f be real-valued, Lebesgue measurable functions on the interval $[0, 1]$. Prove, or provide a counterexample, to each of the following:

(a) If $f_n \to f$ in L^1, then $f_n \to f$ almost everywhere.

(b) If $f_n \to f$ in measure, then $f_n \to f$ in L^1.

(c) If $f_n \to f$ almost uniformly, then $f_n \to f$ almost everywhere. (Note: $f_n \to f$ *almost uniformly* if, for each $\epsilon > 0$, there exists a set $E \subset [0, 1]$ with $m(E) < \epsilon$ so that on $[0, 1] \setminus E$, $f_n \to f$ uniformly.)

Problem 3

Let $(X, \mathbb{A}, \mu)$ be any sigma-finite measure space, and $\{A_n\}_{n=1}^{\infty}$ a sequence of measurable subsets of X. Show that:

(a) $\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$.

(b) Let $\limsup A_n = \bigcap_{n \geq 1} \bigcup_{n \geq k} A_n$. Show that $\mu(\limsup A_n) \geq \limsup \mu(A_n)$.

Problem 4

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be monotone increasing. Show that $\{x : f \text{ is not continuous at } x\}$ is countable.

(b) Show that if $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then f is of bounded variation.

(c) Give an example of a function $f : [0, 1] \to \mathbb{R}$ which is monotone, of bounded variation, and not absolutely continuous. Justify!

Problem 5

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$. Prove the equivalence of the following properties of an orthonormal set $\{e_k\}_{k=1}$ in $\mathcal{H}$.

(i) Finite linear combinations of elements in $\{e_k\}_{k=1}$ are dense in $\mathcal{H}$.

(ii) If $f \in \mathcal{H}$ and $(f, e_j) = 0$ for all j, then $f = 0$.

(iii) If $f \in \mathcal{H}$, and $S_N(f) = \sum_{k=1}^{N} a_k \, e_k$, where $a_k = (f, e_k)$, then $S_N(f) \to f$ as $N \to \infty$ in the norm.

(iv) If $a_k = (f, e_k)$, then $\|f\|^2 = \sum_{k=1}^{\infty} |a_k|^2$.

Problem 6

(a) Define the orthogonal complement of a closed subspace of a Hilbert space.

(b) Prove that if S is a closed subspace of a Hilbert space $\mathcal{H}$ then $\mathcal{H} = S \bigoplus S^{\perp}$, $S^{\perp}$ denotes the orthogonal complement of S.

(c) If S is a subspace of a Hilbert space, is it true that $(S^{\perp})^{\perp} = S$? Explain your answer.

Problem 7

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$. Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear transformation. Prove that there exists a unique bounded linear transformation T^* on $\mathcal{H}$ satisfying the following properties:
(i) $(Tf, g) = (f, T^*g)$,
(ii) $\|T\| = \|T^*\|$, and
(iii) $(T^*)^* = T$.

Problem 8

All questions in this problem refer to Lebesgue measure.

(a) Is $L^1[0, 1]$ contained in $L^2[0, 1]$? Is $L^2[0, 1]$ contained in $L^1[0, 1]$? Justify!

(b) Is $L^1(0, \infty)$ contained in $L^2(0, \infty)$? Is $L^2(0, \infty)$ contained in $L^1(0, \infty)$? Justify!